Nuclear applications of inverse scattering, present ... and future?

R.S. Mackintosh

Department of Physics and Astronomy
The Open University
MK7 6AA, UK

Abstract

There now exists a practical method (IP) for the routine inversion of S-matrix elements to produce the corresponding potential [1]. It can be applied to spin-$\frac{1}{2}$ spin-1 projectiles. We survey the ways that the method can be applied in nuclear physics, by inverting S_{lj} derived from theory or from experiment. The IP method can be extended to invert $S_{lj}(E)$ over a range of energies to produce a potential $V(r,E) + \mathbf{l} \cdot \boldsymbol{\sigma} V_{ls}(r,E)$, yields parity-dependent potentials between pairs of light nuclei [2] and can be convoluted with a direct search on the S-matrix to produce 'direct data $\rightarrow V$ inversion'. The last is an economical alternative form of optical model search to fit many observables (e.g. for polarized deuterons) for many energies, producing an energy-dependent potential with many parameters (e.g. T_R for deuterons) [3].

1   Introduction

It is very easy to derive a cross section or an S-matrix from a potential; the reverse is much harder, but can now routinely be achieved for a wide range of cases. Here we introduce the IP (iterative-perturbative) inverse scattering method, with the emphasis on its range of applicability, illustrated by a few diverse successful applications. We hope to inspire applications that we have not thought of.

Alternative methods exist for some tasks to which IP inversion can be applied (e.g. conventional OM searches to obtain potentials from scattering data, and weighted trivially equivalent potentials (TELPs) to derive DPPs) but IP inversion not only has advantages in these cases but also gives reliable results where no other techniques are available.

2   Inverse scattering: problems and solutions

'Inverse scattering' usually refers to the derivation of the potential corresponding to given S-matrix elements or phase shifts; it may be contrasted with the trivial forward case. This is Case 1 of Table 1. We shall say little about Case 2 of the table, but we will mention Case 3, direct inversion from observables to potential,

since the IP method can readily be convoluted with Case 2 inversion to yield a very powerful Case 3 alternative to conventional optical model searches.

Case 1	forward	$V(r) \to S_l$	EASY
	inverse	$S_l \to V(r)$	MUCH HARDER
Case 2	forward	$S_l \to \sigma(\theta)$	TRIVIAL
	inverse	$\sigma(\theta) \to S_l$	OFTEN VERY HARD
Case 3	forward	$V(r) \to \sigma(\theta)$	EASY
	inverse	$\sigma(\theta) \to V(r)$	VERY HARD

Table 1: Traditional characterization of forward and inverse scattering cases. The σ symbolizes all observables (analyzing powers, etc); S_l includes S_{lj} etc; V includes spin-orbit and tensor terms.

2.1 Traditional methods of $S_l \to V(r)$ inversion

Formal inversion methods apply to two classes:

1. **Fixed-l inversion.** (Gel'fand and Levitan; Marchenko) $S_l \to V(r)$ for S_l for a single l for *all* energies, yielding a local potential.

2. **Fixed energy inversion.** (Newton-Sabatier (NS); Münchow and Scheid (MS)) $S_l \to V(r)$ given S_l for *all* l at a single energy. Related procedures have been developed by Lipperheide and Fiedeldey.

There also exist semi-classical methods for fixed-energy inversion based on WKB and Glauber (eikonal) methods. Practical versions of NS, MS, can handle a finite range of l. The review [1] gives comprehensive references to the inversion techniques.

Disadvantages of traditional methods:
(1) The NS method requires highly precise S_l, and there is a tendency to instability.
(2) The Marchenko method requires large energy ranges, but nuclear potentials are energy-dependent (but it has been used for nucleon-nucleon scattering).
(3) They are not adaptable to cases with small ranges of l (NS).
(3) They mostly apply to spin 0; NS can handle spin 1/2.
(4) They are not readily generalizable.
Nevertheless, there have been a number of applications which have yielded real physical insights, see Ref. [1].

2.2 Information on nuclear interactions from inversion

Inversion can be applied in three distinct general ways:

I. Inversion of S_l, S_{lj} or $S_{ll'}^j$ obtained from theory:

1. Derive dynamic polarization potentials (DPPs) arising from (i) Inelastic scattering, (ii) Breakup processes, (iii) Reaction channels, etc.

2. Derive potential from RGM and similar S-matrix elements.

3. Determine local potentials S-matrix equivalent to non-local potentials.

4. Obtain a potential representation of impulse approximation S-matrix, or the S-matrix from Glauber model and other impact parameter models.

II. Inversion of S_l, S_{lj} or $S_{ll'}^j$ from analysis of experiment:

1. (When there are few partial waves.) Invert S_l from parameterized R-matrix or effective range fits at low energies. (Requires 'mixed case' or 'energy dependent' inversion, see Section 3.3)

2. (When there are many partial waves) High energy two-step phenomenology. (E.g. ^{11}Li, ^{12}C $+^{12}$C from 140 to 2400 MeV)

III. Direct observable $\rightarrow V(r)$ inversion. S-matrix search can be convoluted with IP inversion yielding the S-matrix is byproduct. Many energies can be treated simultaneously to give a multi-component $V(E)$.

3 The Iterative-Perturbative (IP) method

3.1 The key idea

The response of the elastic scattering S-matrix to small changes is assumed to be linear (this is often surprisingly accurate):

$$\Delta S_l = -\frac{\mathrm{i}m}{\hbar^2 k} \int_0^\infty (u_l(r))^2 \Delta V(r) \mathrm{d}r. \tag{1}$$

where $u_l(r)$ is normalized with $u_l(r) \rightarrow I_l(r) - S_l O_l(r)$, I_l and O_l are incoming and outgoing Coulomb wavefunctions. Eq. 1 is readily generalized for spin.

3.2 An Outline of the IP method:

Take a known 'starting reference potential', SRP, $V(r)$ giving S_l. With added term:

$$V(r) \rightarrow \hat{V}(r) = V(r) + \sum c_i v_i(r) \tag{2}$$

it gives $S_l + \Delta S_l$. Functions $v_i(r)$ belong to a suitable 'inversion basis'.

The core of the method is the solution using SVD of the over-determined linear equations derived from Eqn. 1 with $\Delta S_l = S_l^{\text{target}} - S_l$ and $\Delta V = \sum c_i v_i(r)$ to find amplitudes c_i such that $\hat{V}$ gives S_l closer to S_l^{target}. By iterating the linear equations, $S_l + \Delta S_l$ converges to S_l^{target}. There is often a natural starting potential; it can often be zero, or the 'bare potential' when establishing DPP contributions.

The facility of the IP method to control the fitting is a key element in the method, especially in view of the innate ill-posedness (Ref. [4]) of the inversion problem; it is possible always to demand smooth potentials. This and all aspects are fully discussed in Ref. [1] with many references.

3.3 Generalizing from fixed-energy inversion

IP Inversion can be generalized indefinitely as the following progression suggests:
1. Fixed-energy inversion S_l, 'all l, one E' inversion. However, at low energies the potential is generally under-determined, there being too few active partial waves to define the required potential.
2. Mixed case (energy bite) inversion. The problem of under-determination at low energies can often be solved given $S_l(E)$ over a range of energies ('energy bite'). This is 'some l, some E', $S_l(E) \to V(r)$, or 'mixed case', inversion. For a narrow energy bite, this is effectively includes dS_l/dE as input information.
3. Energy dependent inversion. Nuclear potentials, particularly the imaginary parts, vary with energy, but IP inversion can be extended to determine $V(r, E)$ directly: 'some l, some E', $S_l(E) \to V(r, E)$. To date, cases have been limited to the factored form $\sum_i f_i(E) V_i(r)$, where i indicates real-central, imaginary-central, spin-orbit, Majorana, etc.
4. Inversion to fit bound state and resonance energies. Energies of bound states can be included with S_l as input information to determine V.
5. Direct data to potential inversion. Example: $\vec{d} - {}^4\text{He}$, multi-energy.

IP inversion can be applied to the case of identical bosons where only even partial waves are involved, e.g. ${}^{12}\text{C} + {}^{12}\text{C}$. It also applies to the case, very important with light nuclei, where the potential is parity dependent. Such a potential can either be represented as a sum of independent even-parity and odd-parity terms, or as a sum of Wigner (W) and Majorana (M) terms:

$$V_\text{W}(r) + (-1)^L V_\text{M}(r).$$

It is always found that even-parity and odd-parity potentials have different radial forms which often implies a surface peaked Majorana term. Both RGM S-matrices and experimental data imply that there is significant parity dependence even for nucleons on nuclei as heavy as ${}^{16}\text{O}$.

3.4 Spin cases that can be handled by IP inversion

1. Spinless projectiles. $S_l \to V(r)$.
2. Spin 1/2 projectiles. $S_{lj} \to V(r) + \mathbf{l} \cdot \sigma V_{ls}(r)$.
3. Spin one projectiles. Vector spin-orbit and $T_\text{R} \equiv ((\mathbf{s} \cdot \hat{\mathbf{r}})^2 - 2/3) V_R(r)$ tensor potentials can be determined from non-diagonal $S_{ll'}^j$. This is *coupled channel inversion*.
4. High channel spin. For cases like $\text{d} + {}^3\text{He}$, independent potentials for each possible channel spin have been determined.

In every case, all spin-dependent components may have real and imaginary and Wigner and Majorana terms. These can all be expanded in different bases.

3.5 How well does it work?

Fig. 1 shows a test case [5] in which a $S_{ll'}^j$ for deuterons on ${}^{58}\text{Ni}$ at 56 MeV and a known potential, including a tensor term, were inverted with an arbitrary SRP. IP inversion can be applied to noisy data and produce meaningful interactions because

the departure of the final potential from the 'starting potential' of the iterative method is under control, see Section 4.3.
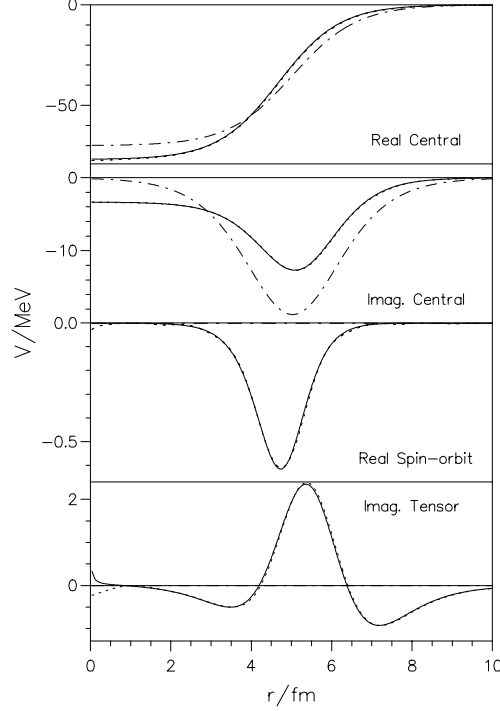

Figure 1: Deuterons on ^{58}Ni at 56 MeV. Solid lines: target (known) potential; dots: V found by inversion; dash-dot: inversion SRP (starting potentials, zero for the real spin-orbit and tensor terms.)

4   Selected applications of IP inversion

Light nuclei, parity dependence.   Scattering between various pairs of light nuclei have been studied by inverting S-matrices from RGM calculations and from R-matrix fits to experimental data, over a wide range of energies. These studies reveal the importance of a parity-dependent (Majorana) component. Ref. [1] has references to parity-dependent potentials for various pairs of light nuclei and presents a case study of $p+^4$He, comparing potentials from empirical and theoretical S_{lj}. The contrasting Majorana potential for $d+^4$He is discussed in Ref. [6] and that for $p+^6$He in Ref. [2].

The dynamic polarization potential, DPP. It is well known that the coupling to breakup channels generates a repulsive DPP for projectiles such as ^{6}Li and ^{2}H. Inverting the elastic scattering S-matrix from a coupled channel calculation, and subtracting the bare potential, gives a local-equivalent l-independent representation of the DPP. The form of the DPP depends on the L-transfer in a systematic way [7]. Nucleus-nucleus interaction also receive large contributions from coupling to inelastic and (especially) reaction channels that *cannot be represented by renormalizing folding model potential*. Many cases are described in Ref. [1], and the contribution

of breakup to the $p+^6$He interaction is presented in Ref. [8], revealing the limitations of folding models for halo nuclei.

Potentials from empirical data. Elastic scattering potentials, including ^{12}C + ^{12}C, ^{16}O + ^{16}O, ^{11}Li scattering and nucleon scattering, have been determined either by fitting S to data and then inverting, or by using the 'direct inversion' in which the S-matrix search is convoluted with the $S \rightarrow V$ inversion. The definitive phenomenology for $p+^{16}$O has been carried out in this way [9], revealing, *inter alia*, the necessity for a Majorana term. Direct inversion of multi-energy data for scattering of light nuclei provides an alternative means of establishing phase shifts $\delta_{lj}(E)$ that behave in a way that is consistent with a potential [10] that varies smoothly with energy. Inversion of S_{lj} from R-matrix fits also has this property.

4.1   Pickup coupling effect in ^{8}He(p, p) elastic scattering

It known that pickup coupling ($p \rightarrow d \rightarrow p$ for proton scattering) makes a significant contribution to the nucleon optical potential. This is one reason that precise fits for nucleon elastic scattering below 50 MeV for closed shell target nuclei have not been found with conventional optical model fitting (for non-closed shell nuclei it is easier to find parameters that fit the shallower diffraction minima). It is now possible to do full finite-range pickup calculations including non-orthogonality terms, and these have been carried out [11] for the p-^{8}He system. The coupling has a large effect on the elastic scattering angular distributions. In Table 2 we quantify the pickup contribution in terms of volume integrals. The repulsive real DPP is quite large at the nuclear centre although the effect on the volume integral is modest. The radial form of the DPP could *not* be represented by renormalizing a folding model potential. The volume integrals reveal the importance of including the non-orthogonality correction.

	$J_{\rm R}$	$\langle r^2 \rangle_{\rm R}^{1/2}$	$J_{\rm I}$	$\langle r^2 \rangle_{\rm I}^{1/2}$	$J_{\rm SOR}$	$J_{\rm SOI}$
OM	704.14	3.092	55.37	3.336	26.60	0.005
CRC	653.94	2.938	307.47	4.138	40.27	1.25
NONO	571.28	2.840	252.62	4.360	33.15	6.55

Table 2: For 15.6 MeV protons on ^{8}He, volume integrals per nucleon pair/(MeV fm^3), and rms radii/fm of the bare potential (OM) and the potentials found by inversion for the complete CRC calculation and for that in which non-orthogonality term was omitted (NONO).

4.2   The DPP due to breakup for ^{6}He scattering from ^{208}Pb.

The breakup for this case [12, 13] was calculated using the same model for ^{6}He as Ref. [8]. However the DPP is now very different, having a long range attractive tail generated by the Coulomb dipole interaction. Fig. 2 shows the short range repulsive/emissive DPP in the surface region. Local regions of emissiveness are a common feature of local potentials representing the highly non-local and l-dependent dynamical polarization contributions; unitarity is not broken. The DPP is not well-defined for $r \leq 10.5$ fm.
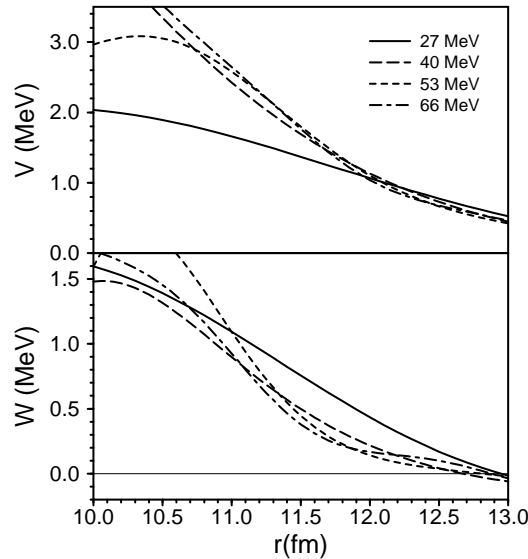
Figure 2: For ^{6}He incident on ^{208}Pb, 27 to 66 MeV, the DPP at the nuclear surface.

Fig. 3 shows the long range attractive and absorptive potentials generated by the dipole coupling; the real part is appreciable out to 60 fm. Both the real and imaginary DPPs strongly influence the elastic scattering differential cross section.

4.3   The d-^{4}He interaction derived from multi-energy data

'Direct data $\rightarrow V$ inversion' is an alternative [10] to optical model fitting with parameterized forms for determining potentials from data, especially when there are many data and many parameters. This is the case for d-^{4}He scattering when there is a full set of polarization observables (including all 3 tensor analyzing powers) for many angles, all for many energies ranging from 4 to 13 MeV. Refs. [3] fitted 1000 data points (five observables, a wide angular range and many energies) to produce a multi-component (Wigner and Majorana, central, spin-orbit and tensor) multi-energy potential (components were functions of energy) giving a reasonable representation of shape resonances. This would have been a formidable task for standard optical model codes since they would have had to include the coupled channel calculation for the tensor interaction within the search.

5   Possible future applications

1. Systematic CRC calculations followed by inversion of the resulting elastic S_{lj} provides a method for establishing shell corrections to the nucleon OM potential.
2. There exists much data for the elastic scattering of neutrons from ^{12}C. Inversion of phase shifts fitted to this data would yield a potential model fitting the non-resonant scattering, allowing an extrapolation to higher neutron energies. Comparing the potential derived in this way with theoretical and standard empirical OM potentials would support (or the opposite) the neutron scattering data as well as
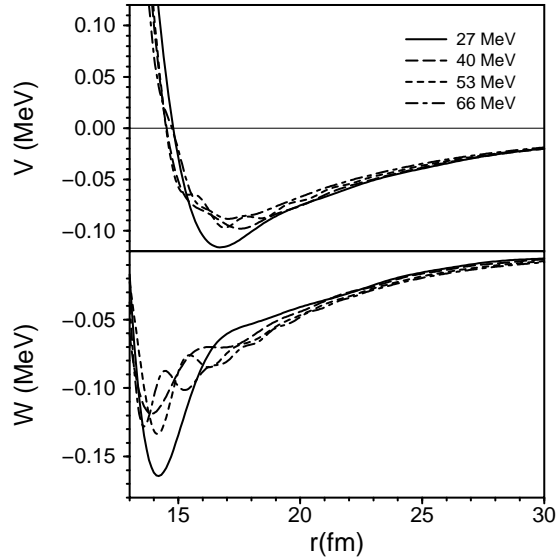
7

Figure 3: For ^{6}He incident on ^{208}Pb, 27 to 66 MeV, the DPP outside the nucleus.

the theoretical models.

3. The IP algorithm appears to be indefinitely generalizable, and has found wider applications than originally envisaged, so are there new applications to nuclear data evaluation? Any suggestions?

References

[1] V.I. Kukulin and R.S. Mackintosh, J. Phys. G: Nucl. Part. Phys. **30** (2004) R1.

[2] R.S. Mackintosh, Nucl. Phys. **A 742** (2004) 3.

[3] S.G. Cooper and R.S. Mackintosh, Nucl. Phys. **A723** (2003) 45.

[4] H. Leeb, private communication.

[5] S.G. Cooper, V.I. Kukulin, R.S. Mackintosh and V.N. Pomerantsev, Nucl. Phys. **A677** (2000) 187.

[6] R.S. Mackintosh, Nucl. Phys. **A 728** (2004) 415.

[7] A.A. Ioannides and R.S. Mackintosh, Phys. Lett. **169B** (1986) 113.

[8] R.S. Mackintosh and K. Rusek, Phys. Rev. **C 67** (2003) 034607.

[9] S.G. Cooper, Nucl. Phys. **A 618** (1997) 87.

[10] S.G.Cooper, V.I. Kukulin, R.S. Mackintosh and E.V. Kuznetsova, Phys. Rev. **C 58** (1998) R31.

[11] F. Skaza, N. Keeley, *et al*, Phys. Lett. **B 619** (2005) 82.

[12] R.S. Mackintosh and N. Keeley, Phys. Rev. **C70** (2005) 024604.

[13] N. Keeley and R.S. Mackintosh, Phys. Rev. **C71** (2005) 057601.